

K-theoretic Gap labeling for Quasicrystals

Fonger Ypma

ABSTRACT. We will review the use of noncommutative topology in the generalisation of Bloch theory from crystals to quasicrystals. After introducing Bloch theory, we will construct the noncommutative space of tilings and we will argue that this is the noncommutative analogue of the Brillouin zone which is used in Bloch theory. The K-theory of the noncommutative Brillouin zone will be used to provide a labeling of the gaps in the spectrum of quasiperiodic Hamiltonians, which can be seen as first step towards a generalisation of Bloch theory to quasicrystals.

1. Introduction

Quasicrystals, as discovered by Schechtman *et. al.* [25] in 1984, form an intermediate phase between periodically ordered crystals and unordered amorphous metals. They are characterised by their diffraction patterns, which are not blurred like those of amorphous metals, but show sharp Bragg peaks just as in the case of crystals. The crucial difference with crystals is that the rotational symmetry in the diffraction spectra of quasicrystals is incompatible with periodicity, which means that in quasicrystals there is no lattice that acts as a group of translational invariance. Likewise, there is no Brillouin zone. This means that the usual methods of Bloch theory to compute the spectrum of one-electron Hamiltonians fail. In general, quasiperiodic Hamiltonians will not have a band spectrum, and the computation of the spectrum becomes very hard. However, by replacing the symmetry group by a groupoid, and the Brillouin zone by a noncommutative C^* -algebra, it turns out that using the K-theory of this C^* -algebra, one can get a labeling of the gaps in the spectrum of the Hamiltonian. This might give qualitative results about the spectrum.

2000 *Mathematics Subject Classification.* Primary 52C23, 46L80; Secondary 19K14, 46L60, 46L85, 82D25.

This article is a summary of my MSc thesis [29]. It has been written for the proceedings of the Summer School on "Topological and Geometric Methods in Quantum Field Theory" in July 2004 in Villa de Leyva (Colombia). I would like to thank the organisers of the school and the other participants for a very stimulating and enjoyable three weeks. Moreover, I would like to thank Klaas Landsman for his encouraging supervision of my MSc thesis, Michelangelo Vargas for various interesting discussions, Jean Bellissard and Johannes Kellendonk for valuable comments, and my current supervisor Keith Hannabuss for carefully reading the manuscript.

©0000 (copyright holder)

The use of noncommutative geometry in solid state physics was pioneered in the eighties by J. Bellissard. His description of the Integer Quantum Hall effect (see [10] for a review) is well known. There is also much work on gap labeling theorems for aperiodic Schrödinger operators (see for instance [5, 8, 7, 9]). To construct the noncommutative Brillouin zone, we model the quasicrystal by a tiling, and construct the "hull" Ω as the space of all translates of this tiling endowed with a certain metric, following the approach of J. Kellendonk [17, 18]. This construction resembles Connes' construction of the space of Penrose tilings [13], but is more general. Next, we use the dynamical system given by the action of the translation group \mathbb{R}^d on Ω to define the transformation groupoid $\Omega \rtimes \mathbb{R}^d$. The convolution algebra of continuous functions on $\Omega \rtimes \mathbb{R}^d$ with compact support can be completed to a C*-algebra, which will be the noncommutative analogue of the Brillouin zone. Finally, we will show why the K-theory of this C*-algebra serves as gap labeling for a Hamiltonian which has the same quasiperiodicity as the underlying tiling.

2. Bloch theory

The motion of electrons in a medium is governed in the independent electron approximation by a one electron Hamiltonian $H = -\nabla^2 + V$ acting on the Hilbert space $L^2(\mathbb{R}^d)$, where $V \in L^\infty(\mathbb{R}^d)$ is a potential that depends on the Coulomb interaction the electron has with ions and other electrons. In a crystal, we will assume that the potential has the periodicity of a lattice Γ , i.e. $V(x+a) = V(x)$ for all $a \in \Gamma$. To compute the spectrum of H , we note that the Hamiltonian H commutes with all translation operators $T(a)$ given by $T(a)\psi(x) = \psi(x+a)$, where $\psi \in L^2(\mathbb{R}^d)$. Note that T can be seen as the (right) regular representation of Γ . Since Γ is locally compact, by the Stone-Naimark-Ambrose-Godement Theorem [3] this representation can be decomposed as a direct integral of irreducible representations, and since Γ is abelian, these are just the characters $\hat{\Gamma}$ of Γ (see also [5, 22]). In this case, $\hat{\Gamma}$ will be a torus in reciprocal space, called the **Brillouin zone** in solid state physics [1]. We get the decomposition

$$(2.1) \quad T(a) = \int_{\hat{\Gamma}}^{\oplus} e^{ik \cdot a} d^d k,$$

and likewise, the Hilbert space $L^2(\mathbb{R}^d)$ on which $T(a)$ acts can be decomposed as

$$L^2(\mathbb{R}^d) \simeq \int_{\hat{\Gamma}}^{\oplus} \mathcal{H}_k d^d k.$$

Elements $\psi \in L^2(\mathbb{R}^d)$ can thus be seen as sections $\psi : \hat{\Gamma} \rightarrow \{\mathcal{H}_k\}$ such that $\psi_k \in \mathcal{H}_k$, and it follows from the decomposition (2.1) that every $\psi_k \in \mathcal{H}_k$ obeys

$$(2.2) \quad \psi_k(x+a) = T(a)\psi_k(x) = e^{ik \cdot a} \psi_k(x).$$

A norm on each \mathcal{H}_k is given by $\|\psi_k\|^2 = \int_{\mathbb{R}^d/\Gamma} |\psi_k(x)|^2 d^d x < \infty$. Note that elements of \mathcal{H}_k will in general not belong to $L^2(\mathbb{R}^d)$, since their L^2 -norm will be infinite. Using the fact that the Hamiltonian commutes with all the $T(a)$'s, we can decompose it as a direct integral,

$$H = \int_{\hat{\Gamma}}^{\oplus} H_k d^d k,$$

where each H_k acts on \mathcal{H}_k . For each $k \in \hat{\Gamma}$ fixed, the spectrum of H_k is easy to compute because of the "boundary conditions" (2.2): it is discrete and bounded from below [22]. The spectrum of the Hamiltonian H is given by

$$\sigma(H) = \cup_{k \in \hat{\Gamma}} \sigma(H_k).$$

Note that the (improper) eigenfunctions $\psi_{k,n}$ of H obey (2.2), and are thus extended wavefunctions.

The above is a summary of Bloch theory for periodic media. It is clear that the role of the lattice Γ and its Pontryagin dual, the Brillouin zone $\hat{\Gamma}$, is very important.

3. The noncommutative Brillouin zone

In order to be able to compute spectra of quasiperiodic Hamiltonians, we have to generalise Bloch theory to quasicrystals. A quasiperiodic potential $V \in L^\infty(\mathbb{R}^d)$ can be approximated by trigonometric polynomials, with n pairwise incommensurate frequencies, where $n > d$. In the quasiperiodic case, there is no lattice Γ nor a Brillouin zone $\hat{\Gamma}$. To overcome this problem, it turns out that one should replace the symmetry group Γ by a transformation groupoid, and the Brillouin zone by the corresponding groupoid C*-algebra, which can be seen as a noncommutative Brillouin zone. Following [18], we will first construct the **hull** Ω as space of tilings, then the groupoid $\Omega \times \mathbb{R}^d$, and finally we will construct its C*-algebra. We will indicate why this is the noncommutative analogue of the Brillouin zone. Note that one can also construct the hull in terms of translates of the resolvent of H [5] or in terms of Delone sets [6, 8].

Physically, a quasicrystal can be modeled by a d -dimensional **tiling** T , i.e. a countable set of closed subsets t_i of \mathbb{R}^d , each of positive Lebesgue measure, such that their interiors don't overlap. We will assume that there is a finite number of **prototiles** (i.e. equivalence classes of tiles under congruence modulo translation), and that every tile is homeomorphic to the closed unit ball. An example is a periodic tiling with lattice of periodicity Γ , where every tile is a unit cell, homeomorphic to \mathbb{R}^d/Γ . Quasiperiodic tilings can be constructed by projecting an n -dimensional periodic tiling down to a d -dimensional subspace in a certain way (the "projection method") or by the so called substitution method. The most famous quasiperiodic example is given by the Penrose tiling. For more on tilings, see [15].

To model the motion of an electron in a quasicrystal (or its tiling T), we change to the reference frame of the electron, and move the tiling around. Thus, we look at the set $\{T + x \mid x \in \mathbb{R}^d\}$ of all translates of a given tiling T . Obviously, every element of this set is itself a tiling of \mathbb{R}^d . We endow this set with a metric d given by

$$d(T_1, T_2) = \inf \left\{ 0 \leq \epsilon < 1 \mid \exists |x_1|, |x_2| < \epsilon : (T_1 + x_1) \upharpoonright_{B(0, \frac{1}{\epsilon})} = (T_2 + x_2) \upharpoonright_{B(0, \frac{1}{\epsilon})} \right\}.$$

If there is no such $\epsilon < 1$, we set $d(T_1, T_2) = 1$. In other words, tilings are "close" to each other if they coincide on a large patch (with radius $1/\epsilon$) around the origin (up to a small displacement vector). Next, we define the **hull** Ω_T as the completion of the metric space $(\{T + x \mid x \in \mathbb{R}^d\}, d)$. For a periodic tiling by unit cells with lattice Γ , Ω_T is homeomorphic to \mathbb{R}^d/Γ . For quasicrystals, the hull is a compact space, and is independent of the choice of the initial tiling $T \in \Omega_T$. This follows from certain repetitivity conditions that hold for quasiperiodic tilings [18]. Hence, the reference to T in Ω can be omitted. Note that if the tiling is not repetitive,

the metric d is not a canonical one, as it depends on the choice of the origin. In these cases, it is better to use a more canonical method. However, for models of quasicrystals, the above metric suffices.

REMARK 3.1. Observe that Ω/\mathbb{R}^d is a non-Hausdorff space with trivial topology from a commutative point of view. However, its noncommutative topological structure, captured in the groupoid C^* -algebra defined below, is far from being trivial. This is the philosophy Connes uses for his space of Penrose tilings [13]. However, his construction only works for substitution tilings, while Kellendonk's approach is more general. Connes' space of index sequences modulo finite differences can be seen as space of Penrose tilings modulo isometries (instead of modulo translations) [29].

We use the group action of \mathbb{R}^d on Ω to define the groupoid $\Omega \rtimes \mathbb{R}^d$. Recall that a **groupoid** G is defined as a small category in which every morphism is invertible. Alternatively, it can be seen as a generalisation of a group in which the product is only defined on a subset of $G \times G$. The set of objects G^0 of the category can be seen as space of units. A groupoid comes with target and source maps $t, s : G \rightarrow G^0$; The product of $(x, y) \in G \times G$ is defined if and only if $t(y) = s(x)$.

An example of a groupoid is the groupoid G of paths in a space X : the unit space will be identified with X , and the paths are composable if and only if the endpoint of the first path (its "target") is the starting point of the second path (its "source"). Obviously, a group is a groupoid, with space of units consisting only of the unit element e of the group. Another example of a groupoid is the graph $R = \{(x, x') \mid x \sim x'\}$ of an equivalence relation \sim ; the product is given by $(x, y) \cdot (y, z) = (x, z)$, and the inverse by $(x, y)^{-1} = (y, x)$; note that these are dictated by the transitivity and symmetry properties of the equivalence relation. For more on groupoids, see for instance [21, 23].

The basic example we will be concerned with is the transformation groupoid $\Omega \rtimes \mathbb{R}^d$: elements are given by pairs $(T, x) \in \Omega \times \mathbb{R}^d$, with product and inverse given by

$$(T, x)(T - x, y) := (T, x + y); (T, x)^{-1} := (T - x, -x).$$

It is a locally compact groupoid in the product topology of $\Omega \times \mathbb{R}^d$. Note that this groupoid is isomorphic and homeomorphic to the equivalence relation groupoid

$$R_T := \{(T, T') \in \Omega \times \Omega \mid \exists x \in \mathbb{R}^d \text{ such that } T = T' + x\}$$

if and only if Ω contains no periodic tilings [18, 29].

To construct the noncommutative Brillouin zone, we define the groupoid C^* -algebra of $\Omega \rtimes \mathbb{R}^d$ as follows (see [21, 23] for more on groupoid C^* -algebras). Consider the set $C_c(\Omega \rtimes \mathbb{R}^d)$ of continuous functions with compact support, and endow it with a convolution product

$$(3.1) \quad f * g(T, x) := \int_{\mathbb{R}^d} f(T, y)g(T - y, x - y)dy,$$

and involution $f^*(T, x) = \overline{f(T - x, -x)}$. Note that since the fibers $s^{-1}(T)$ and $t^{-1}(T)$ can be identified with \mathbb{R}^d for each $T \in \Omega$, the left Haar system [23] of measures on the groupoid reduces to the usual Lebesgue measure on \mathbb{R}^d . Next, we define reduced representations π_T of this $*$ -algebra on the Hilbert space $L^2(\mathbb{R}^d)$ by

$$(3.2) \quad \pi_T(f)\psi(x) := \int_{\mathbb{R}^d} f(T - x, y - x)\psi(y)dy.$$

Under the above mentioned repetitivity conditions for T , it is easy to show that every representation π_T is faithful. Now a norm on $C_c(\Omega \rtimes \mathbb{R}^d)$ will be given by $\|f\| := \sup_{T \in \Omega} \|\pi_T(f)\|_{L^2(\mathbb{R}^d)}$, and the completion in this norm is the C*-algebra $C^*(\Omega \rtimes \mathbb{R}^d)$, which we will call A_T . Recall that in the periodic case, with lattice Γ , the hull is homeomorphic to \mathbb{R}^d/Γ . Applying a theorem of Green (see [14, 19]) to this situation, one can show that

$$(3.3) \quad C^*(\mathbb{R}^d/\Gamma \rtimes \mathbb{R}^d) \simeq C(\hat{\Gamma}) \otimes \mathcal{K}(L^2(\mathbb{R}^d/\Gamma)),$$

where \mathcal{K} denotes the compact operators. In other words, the groupoid C*-algebra is Morita equivalent to the C*-algebra of continuous functions on the Brillouin zone $\hat{\Gamma}$. This is the motivation to call the C*-algebra $A_T = C^*(\Omega \rtimes \mathbb{R}^d)$ the **noncommutative Brillouin zone**. Note that a similar construction can be applied to the groupoid R_T . The C*-algebra A_T is sometimes also called the "C*-algebra of observables". We stress that the transition from commutative to noncommutative Brillouin zone coincides with the transition from periodic to quasiperiodic structures.

REMARK 3.2. Physically, the (continuous) motion of an electron in a quasicrystal (or on the tiling that models the quasicrystal) can be approximated by a discrete motion, where the electron is considered to be "hopping" from tile to tile. In the resulting "tight binding approximation" [1, 4, 6], the Hamiltonian becomes a bounded operator. Following [17, 18], we can mathematically implement this by assigning to each tile t a puncture $x(t)$, such that if two tiles are translates of each other, then their punctures are translates by the same translation vector. We then define the **discrete hull** Ω_{punc} by

$$(3.4) \quad \Omega_{punc} = \{T \in \Omega \mid x(t) = 0 \text{ for some } t \in T\}.$$

The set of all punctures of a given tiling T will be denoted by T^{punc} . Note that for a periodic tiling with lattice Γ , Ω_{punc} can be identified with a point (or with a finite point set, if more than one puncture per unit cell is allowed), whereas T^{punc} can in this case be identified with the lattice Γ .

Ω_{punc} is an "abstract transversal" to the action of \mathbb{R}^d on Ω , in the sense that it is a closed subset of the unit space Ω of the transformation groupoid $\Omega \rtimes \mathbb{R}^d$ meeting every orbit (under the action of \mathbb{R}^d) in Ω [19]. Under the mentioned regularity conditions for T , Ω_{punc} can be shown to be a Cantor set [18]. We can then define the discrete analogues of the groupoids $\Omega \rtimes \mathbb{R}^d$ and R_T , and their C*-algebras. We will denote the discrete analogue of A_T by A_{punc} . Now these C*-algebras are actually Morita equivalent [16], which means for instance that their K-theory is isomorphic. So for our purposes, we can either use the discrete or the continuous description. The discrete version has the virtue that the C*-algebra is unital, and that the Hamiltonian is a bounded operator on $\ell^2(T^{punc})$ in this case.

Next we'll construct a trace on the noncommutative Brillouin zone, which can be seen as noncommutative integration. First, we claim that for quasiperiodic tilings, there exists a unique translation invariant, ergodic probability measure μ on Ω [4, 29]. Next, we define a functional τ_μ on $C_c(\Omega \rtimes \mathbb{R}^d)$ by

$$\tau_\mu(f) := \int_{\Omega} d\mu(T) f(T, 0),$$

which has the properties of a positive trace, i.e. $\tau_\mu(f^*f^*) \geq 0$ and $\tau_\mu(f^*g) = \tau_\mu(g^*f)$ for all $f, g \in C_c(\Omega \rtimes \mathbb{R}^d)$. It is easy to show that τ_μ is faithful. Unfortunately, τ_μ is not bounded, so it cannot be extended to a trace on A_T . However, for the discrete case, the corresponding functional is bounded, so in this case it can be extended to a trace on A_{punc} . Since the discrete and continuous pictures are equivalent what K-theory is concerned, we will speak about a trace on A_T even if τ_μ can strictly speaking not be extended to a trace on A_T .

4. K-theoretic gap labeling

The question arises how the Hamiltonian $H = -\nabla^2 + V$ that models the motion of an electron on a tiling T_0 is related to the hull, its groupoid and its C*-algebra. Associated to $H = H_{T_0}$, we introduce a family $\{H_T\}_{T \in \Omega}$ of Schrödinger operators satisfying the **covariance condition**

$$(4.1) \quad H_{T+x} = T(x)H_T T(x)^{-1}.$$

Note that the representations π_T of (3.2) satisfy a similar covariance condition. We say that a covariant family $\{H_T\}$ is **affiliated to** the C*-algebra A_T if for all $f \in C_0(\mathbb{R})$ the bounded operators $f(H_T)$ can be represented as $\pi_T(h_f)$ for some $h_f \in C^*(\Omega \rtimes \mathbb{R}^d)$ such that the map $h : C_0(\mathbb{R}) \rightarrow A_T; f \mapsto h_f$ is a bounded *-homomorphism.

Now observe that the spectral projections $\chi_{(-\infty, E]}(H_T)$ are bounded functions of the Hamiltonian, and they are continuous if and only if $E \notin \sigma(H_T)$, i.e. if E lies in a gap \mathfrak{g} of the spectrum. Moreover, observe that in this case, $\chi_{(-\infty, E]}(H_T)$ is independent of the value of $E \in \mathfrak{g}$. This means that each gap \mathfrak{g} in the spectrum of H defines a projection $p_{\mathfrak{g}} \in A_T$, or, in the discrete case, in A_{punc} .

Next we introduce the **integrated density of states (IDS)** of $\{H_T\}$ as the function defined by

$$(4.2) \quad N_T(E) := \lim_{\Lambda \uparrow \mathbb{R}^d} \frac{1}{|\Lambda|} \#\{E' \in \sigma(H_{T, \Lambda}) \mid E' \leq E\}.$$

By unique ergodicity, this limit exists uniformly in T if E is a continuity point of the limit. It is independent of the boundary conditions on Λ [20]. This yields the IDS as a nonnegative, nondecreasing function on \mathbb{R} that is constant on gaps in the spectrum of H_T . The IDS actually determines the continuous part of the spectrum.

N_T is constant as a function of T on a dense subset of Ω . Now a covariant family $\{H_T\}$ affiliated to A_T is said to satisfy **Shubin's formula** if for μ -almost all $T \in \Omega$, we have

$$(4.3) \quad N_T(E) = \tau_\mu(\chi_{H \leq E}),$$

where $\chi_{H \leq E} \in A_T$ is the element such that $\pi_T(\chi_{H \leq E}) = \chi_{(-\infty, E]}(H_T)$ for all T . In [4], Shubin's formula was established for tight binding Hamiltonians, whereas in [5] it was proved to hold for continuous Schrödinger operators. We will denote the common value of the IDS's simply by $N(E)$.

Shubin's formula is the link between the K-theory of the noncommutative Brillouin zone A_T and the IDS on the gaps of the spectrum of H : recall that every gap \mathfrak{g} in the spectrum defines a projection $p_{\mathfrak{g}} \in A_T$. Now the K_0 -group of a unital C*-algebra is built up by equivalence classes under unitary transformations of projections in matrices with entries in the algebra, with operation $[p] + [q] = [p \oplus q]$. The resulting semigroup is made into a group by the Grothendieck construction.

For more on K-theory, see e.g. [12, 24]. Note that although A_T is not unital, its K-theory can still be defined, but to avoid technicalities, we'll work with A_{punc} , which is indeed unital. Since A_T and A_{punc} are Morita equivalent, and K-theory is invariant under Morita equivalence [24], there is no loss in doing so. Now the trace τ_μ is invariant under unitary transformations, so it defines a map $\tau_{\mu*} : K_0(A_{punc}) \rightarrow \mathbb{R}$. The image is a countable subset of \mathbb{R} , because A_{punc} is separable [5]. Since the projections $p_{\mathfrak{g}}$ define elements of the K_0 -group, by (4.3) we have [5]

THEOREM 4.1. *Let $\{H_T\}$ be affiliated to A_{punc} . For μ -almost all $T \in \Omega$, the IDS on gaps of the spectrum of H_T , denoted by $N(\mathfrak{g})$, takes values in the (ordered) countable set $\tau_{\mu*}(K_0(A_{punc})) \cap [0, 1]$. The resulting gap labeling is invariant under norm-continuous perturbations, as long as the gaps do not close.*

The last property of invariance is due to the fact that if two projections are homotopic to each other, they are in the same K-theory class [24]. A few remarks are in order. Notice that one could have used the K-theory of A_T instead of that of A_{punc} , since the two C*-algebras are Morita equivalent. To actually compute these K-theory groups, useful tools are the Pimsner-Voiculescu exact sequence and Connes' Thom isomorphism [12]. Note that the gap labeling depends only on the topology of the noncommutative Brillouin zone. In essence, only the structure of the underlying tiling matters, and this structure can in principle be deduced from the diffraction spectrum of the quasicrystal. There is even a theorem, conjectured once by J. Bellissard and proved independently in [7, 11, 16], stating that

$$(4.4) \quad \tau_{\mu*}(K_0(A_{punc})) = \mu(C(\Omega_{punc}, \mathbb{Z})),$$

in other words, the gap labels are given in terms of integer valued functions on the hull (the functional μ is identified with the measure μ via Riesz' representation theorem). The right hand side can be seen as the \mathbb{Z} -module generated by the occurrence probabilities of finite patches in the underlying tiling. Hence, the gap labeling is determined by these occurrence probabilities.

The gap labeling theorem looks as if it is able to provide direct information about the spectrum of the Hamiltonian, just as Bloch theory does in the periodic case. For example, if the set of gap labels is a relatively dense subset of $[0, 1]$, one can argue that the spectrum must be Cantor-like. However, the problem is that $\tau_{\mu*}(K_0(A_T))$ is only the set of *possible* gap labels. Since not every projection in A_T is necessarily a spectral projection of H , it is not at all guaranteed that all these possible gaps actually occur. Some gaps might be closed, and this cannot be detected by K-theory, which is too rough as topological invariant. In fact, in many cases it is even hard to show the spectrum has gaps at all.

An example of a one-dimensional, discrete, quasiperiodic Hamiltonian is the Almost Mathieu operator

$$(4.5) \quad H_{\alpha, \theta, \lambda} \psi(n) := \psi(n+1) + \psi(n-1) + 2\lambda \cos(2\pi\alpha n + \theta) \psi(n).$$

It is easy to see that for rational $\alpha = p/q$, the potential has periodicity q , and the spectrum will thus be a band spectrum, determined by Bloch theory. However, for irrational α the potential becomes quasiperiodic. The set of gap labels is given by $(\mathbb{Z} + \alpha\mathbb{Z}) \cap [0, 1]$. More than twenty years ago, it was conjectured that for all irrational α , the spectrum is Cantor-like (see [26, 27, 28] for reviews). After a series of partial results, this is now proved [2]. The stronger statement that all gaps (except for the middle one at $E = 0$) are actually open, has still not been

proved for all values of α , as far as the author knows. For general quasiperiodic operators, it is difficult to find a mechanism to determine which gaps are actually open.

In conclusion, we have seen that for a generalisation of Bloch theory from periodic to quasiperiodic materials, one needs to use noncommutative topology. Although the K-theoretic gap labeling is a beautiful result from a mathematical point of view, there is still a long way to go to get a physically meaningful theory of electronic properties of quasicrystals.

References

- [1] N. W. Ashcroft and N. D. Mermin, *Solid state physics*, Holt and Rinehart and Winston, 1976.
- [2] A. Avila and S. Jitomirskaya, *The Ten Martini Problem*, arXiv: math.DS/0503363.
- [3] A. O. Barut and R. Raçzka, *Theory of group representations and applications*, PWN-Polish Scientific Publishers, 1977.
- [4] J. Bellissard, *K-theory of C^* -algebras in solid state physics*, Statistical Mechanics and Field Theory: Mathematical Aspects (T. C. Dorlas, N. M. Hugenholtz, and M. Winnink, eds.), Lecture Notes in Physics, no. 257, Springer Verlag, 1986, pp. 99–156.
- [5] ———, *Gap labelling theorems for Schrödinger's operators*, From Number Theory to Physics (M. Waldschmidt, P. Moussa, J. M. Luck, and C. Itzykson, eds.), Springer Verlag, 1993, pp. 538–630.
- [6] ———, *Noncommutative geometry of aperiodic solids*, Geometric and Topological Methods for Quantum Field Theory (A. Cardona, S. Paycha, and H. Ocampo, eds.), World Sci. Publishing, 2003, Proceedings of summer school in Villa de Leyva, pp. 86–156.
- [7] J. Bellissard, R. Benedetti, and J-M. Gambaudo, *Spaces of tilings, finite telescopic approximations and gap-labelling*, Communications in Mathematical Physics (2004), To appear; arXiv: math.DS/0109062.
- [8] J. Bellissard, D. J. L. Herrmann, and M. Zarrouati, *Hull of aperiodic solids and gap labelling theorems*, Directions in Mathematical Quasicrystals (M. B. Baake and R. V. Moody, eds.), CRM Monograph Series, vol. 13, AMS, 2000, pp. 207–259.
- [9] J. Bellissard, J. Kellendonk, and A. Legrand, *Gap-labelling for three dimensional aperiodic solids*, C. R. Acad. Sci. **332** (2001), no. 6, 521–525.
- [10] J. Bellissard, M. Schulz-Baldes, and A. van Elst, *The noncommutative geometry of the quantum Hall effect*, Journal of Mathematical Physics **35** (1994), 5373–5471.
- [11] M.-T. Benamèur and H. Oyono-Oyono, *Gap-labelling for quasicrystals (proving a conjecture by J. Bellissard)*, Operator Algebras and Mathematical Physics (Constanta 2001) (J.M. Combes et al., eds.), 2001, pp. 11–22.
- [12] B. Blackadar, *k-theory for operator algebras*, Springer, 1998.
- [13] A. Connes, *Noncommutative geometry*, Academic Press, 1994.
- [14] P. Green, *The structure of imprimitivity algebras*, Journal of Functional Analysis **36** (1980), no. 1, 88–104.
- [15] B. Grünbaum and G. C. Shephard, *Tilings and patterns*, Freeman, 1989.
- [16] J. Kaminker and I. Putnam, *A proof of the gap labeling conjecture*, Michigan Math. J. **51** (2003), no. 3, 537–546.
- [17] J. Kellendonk, *Noncommutative geometry of tilings and gap labelling*, Rev. Math. Phys. **7** (1995), 1133–1180.
- [18] J. Kellendonk and I. F. Putnam, *Tilings, C^* -algebras, and K-theory*, Directions in Mathematical Quasicrystals (M. B. Baake and R. V. Moody, eds.), CRM Monograph Series, vol. 13, AMS, 2000, pp. 177–206.
- [19] P. S. Muhly, J. N. Renault, and D. P. Williams, *Equivalence and isomorphism for groupoid C^* -algebras*, J. Operator Theory **17** (1987), 3–22.
- [20] L. Pastur and A. Figotin, *Spectra of random and almost-periodic operators*, Grundlehren der mathematischen Wissenschaften, vol. 297, Springer Verlag, 1992.
- [21] A. L. T. Paterson, *Groupoids, inverse semigroups and their operator algebras*, Birkhäuser, 1999.

- [22] M. Reed and B. Simon, *Analysis of operators*, Methods of Modern Mathematical Physics, vol. IV, Academic Press, 1978.
- [23] J. N. Renault, *A groupoid approach to C^* -algebras*, Lecture Notes in Mathematics, vol. 793, Springer Verlag, 1980.
- [24] M. Rørdam, F. Larsen, and N. J. Laustsen, *An introduction to K -theory for C^* -algebras*, London Mathematical Society Student Texts, vol. 49, Cambridge University Press, 2000.
- [25] D. Schechtman, I. Blech, D. Gratias, and J. W. Cahn, *Metallic phase with long-range orientational order and no translational symmetry*, Phys. Rev. Lett. **53** (1984), no. 20, 1951–1953.
- [26] M. A. Shubin, *Discrete magnetic Laplacian*, Comm. Math. Phys. **164** (1994), no. 2, 259–275.
- [27] B. Simon, *Schrödinger operators in the twentieth century*, J. Math. Phys. **41** (2000), no. 6, 3523–3555.
- [28] ———, *Schrödinger operators in the twenty-first century*, Mathematical Physics 2000 (A. Fokas, A. Grigoryan, T. Kibble, and B. Zegarlinski, eds.), Imperial College Press, 2000, pp. 283–288.
- [29] F. Ypma, *Quasicrystals, C^* -algebras and K -theory*, MSc Thesis, University of Amsterdam, 2004.

INSTITUUT VOOR THEORETISCHE FYSICA, UNIVERSITEIT VAN AMSTERDAM, VALCKENIERSSTRAAT
65, 1018 XE AMSTERDAM, NETHERLANDS

Current address: Mathematical Institute, University of Oxford, 24-29 St. Giles', Oxford,
OX1 3LB, United Kingdom

E-mail address: ypma@maths.ox.ac.uk