Univariate Polynomial Solutions of Polynomial Difference equations

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Abstract. The subject of this talk is difference equations of the form $G(P(x-1), ..., P(x-s)) + G_0(x) = 0$, where $G(x_1, ..., x_s)$ is a real polynomial of a total degree $D \geq 2$, and $G_0(x)$ is a polynomial in x. We consider the following problem: given G and G_0, find an upper bound on the degree d of a real-polynomial solution $P(x)$, if such solution exists.
Such bound does not necessarily exist in general. We give an example of a difference equation which is solvable by a polynomial of an arbitrary degree.
However, in this talk we focus on cases where a bound on the degree d of a polynomial solution does exist. To study such cases we apply the notion of a "degree polynomial", i.e. a univariate polynomial for which d is a root. We formulate a sufficient condition under which such polynomial exists. Using this condition, we can give an effective bound on d, for instance, for all difference equations $G(P(x-1), P(x-2), P(x-3)) + G_0(x) = 0$ with quadratic G, and all difference equations $G(P(x), P(x-t)) + G_0(x) = 0$ with G of an arbitrary degree.
To obtain the results above we use the following technique. Firstly, we equate the coefficients at x^{Dd-l} of the left-hand side of the difference equation to zero. Secondly, from these equations, using Newton-Girard identities between elementary and power-sum symmetric polynomials, we obtain a series of equations, in which variables stay for the unknown degree d and values of power-sum symmetric polynomials in the roots of a solution $P(x)$. Thirdly, we formulate the condition under which the dependency on the roots is eliminated. Thus, under this condition we give the degree polynomial.
It is worth to note that the results are applicable to more general difference equations of the form $G(P(x-\tau_1), ..., P(x-\tau_s)) + G_0(x) = 0$, where $0 \leq \tau_1 < ... < \tau_s$ are real numbers.
At the end of this talk we list the remaining open questions, such as if there is a connection between the presented results and Galois theory.